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RADIATIVE TRANSFER BETWEEN TWO CONCENTRIC SPHERES SEPARATED BY AN ABSORBING AND EMITTING GAS*

INGE L. RYHMING†
Aerospace Corporation, El Segundo, California

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Abstract—The transfer of radiant energy in an atmosphere having spherical symmetry is discussed. As an application, the transfer between two concentric spheres (having black surfaces) kept at different uniform temperatures and separated by an absorbing and emitting gas with a constant absorption coefficient is studied. The temperature in the gas along a radius satisfies a Fredholm integral equation of the second kind. Exact solutions are obtained by a numerical method and limiting solutions pertaining to optically thick and thin conditions are given in closed form. The temperature distribution along a radius depends strongly on the temperature ratio of the two surfaces, the absorption coefficient, and the radii of the spheres.

INTRODUCTION

THE TRANSFER of radiative energy in plane-parallel atmospheres has been studied in great detail and reported in the astrophysical literature [1, 2] and in literature dealing with heat transfer [3]. In particular, the problem of radiative transfer between two infinite parallel plates kept at two different temperatures and separated by an absorbing and emitting gas has recently received much attention [4–7].

Problems of transfer in atmospheres with spherical symmetry have received comparatively little attention. However, certain aspects of such problems are discussed by Chandrasekhar [1]. Because transfer in atmospheres with spherical symmetry is important in many applications, a brief treatment is given in the first part of this paper to extend the theory to this case. In this analysis, formulas are given for the specific intensity, the rate of absorption, the rate of heat input to the gas per unit volume, and the rate of radiant flux. The formulas are subsequently applied to the problem of transfer between two concentric spheres (having black surfaces) kept at different uniform temperatures and separated by an absorbing and emitting gas with a constant absorption coefficient. The integral equation for the temperature distribution between the two spheres is solved numerically, and exact limiting solutions are given in closed form.

Problems of this nature can arise in connection with the study of intense explosions. The temperature field created by a sphere of hot gas appearing as a black spherical body and the associated heat-transfer rates are thus of immediate interest.

Subsequent to the completion of the present work, the author received a paper by Sparrow, Usiskin and Hubbard [8] that does, in fact, consider the problem of transfer between two concentric spheres. In their analysis, the spheres have the same temperature, and internal heat generation is assumed in the gas. Therefore, in the problem considered here, different boundary conditions are used; furthermore, the methods of the present paper are formally different from those of Sparrow, Usiskin and Hubbard. A more detailed analysis of the present problem is given in reference [9].

SPECIFIC INTENSITY OF RADIATION IN A SPHERICALLY SYMMETRIC ATMOSPHERE

Consider a spherically symmetric atmosphere consisting of an absorbing and emitting gas characterized by the absorption coefficient $\varkappa_\nu$.

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† Member of the Technical Staff, Aerophysics Department, Aerodynamics and Propulsion Research Laboratory. Present address FFA, Bromma 11, Sweden.

All thermodynamic variables in the atmosphere, such as the temperature and the density, are functions of a single variable r, the radius vector measured from the center of the atmosphere. In general, $\varkappa_\nu$ depends on the thermal state of the gas and is consequently also a function of r. In dealing with the transfer of radiant energy within such an atmosphere, the expression is needed for the specific intensity I_ν (or simply, the *intensity*) of the radiation at a given point r in a given direction. If no incident radiation from the outside falls on the atmosphere, the intensity will be a function of r and the inclination θ to radius vector. In terms of these two variables, the equation of transfer can be written [1]

$$\cos\theta \frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} = -\varkappa_\nu(r)[I_\nu(r,\theta) - B_\nu(r)] \tag{1}$$

By introducing the Cartesian coordinates x and y the integration of equation (1) can be performed along lines of constant y (Fig. 1). At the point r we distinguish now between the intensity in the positive and the negative x-direction, and denote these two intensities by $I_\nu^+(r,\theta)$ and $I_\nu^-(r,\theta)$, respectively (Fig. 1). Furthermore, let us assume that the atmosphere is bounded and does not extend beyond a radius $r = R$. The nature of the boundary need not be considered at this point. It is sufficient only to identify the intensity in the positive x-direction at the point

$$x = -[R^2 - r^2\sin^2\theta]^{\frac{1}{2}}$$

on the boundary with $I_\nu^+(R,\theta)$, and similarly, the intensity in the negative x-direction at the point $x = +[R^2 - r^2\sin^2\theta]^{\frac{1}{2}}$ on the boundary with $I_\nu^-(R,\theta)$. The formal solution to equation (1) for $I_\nu^+(r,\theta)$ can then be written

$$I_\nu^+(r,\theta) = I_\nu^+(R,\theta)\exp[-\tau_\nu(x,-x_R)] + \int_{-x_R}^{x} \varkappa_\nu(\bar r)B_\nu(\bar r)\exp[-\tau_\nu(x,\bar x)]\,\mathrm{d}\bar x \tag{2}$$

where

$$\bar r = [\bar x^2 + r^2\sin^2\theta]^{\frac{1}{2}},\quad x_R = [R^2 - r^2\sin^2\theta]^{\frac{1}{2}},$$

and $\tau_\nu(x,\bar x)$ is the optical depth along $y = r\sin\theta = \text{const.}$ between two points with coordinates $\bar x$ and x. Thus for τ_ν, we have the definition

$$\tau_\nu(x,\bar x) = \int_{\bar x}^{x} \varkappa_\nu(r')\,\mathrm{d}x' \tag{3}$$

where $r' = [x'^2 + r^2\sin^2\theta]^{\frac{1}{2}}$ (Fig. 1). For $I_\nu^-(r,\theta)$ one has a similar formula. The atmosphere is taken to be in local thermodynamic equilibrium; hence, the source function $B_\nu(r)$ in equations (1) and (2) is the Planck function.

It is convenient to choose the radius vector as the integration variable rather than the x-variable occurring in equations (2) and (3). In performing such a transformation, we can write $I_\nu^+(r,\theta)$ in the form

$$\left.\begin{aligned} I_\nu^+(r,\theta) &= I_\nu^+(R,\theta)\exp[-(\eta_\nu + \eta_{\nu R})] \\ &+ \int_{r_o}^{R} \varkappa_\nu(\bar r)B_\nu(\bar r)\exp[-(\eta_\nu + \bar\eta_\nu)]\frac{\bar r\,\mathrm{d}\bar r}{(\bar r^2 - r_o^2)^{\frac{1}{2}}} \\ &+ \int_{r_o}^{r} \varkappa_\nu(\bar r)B_\nu(\bar r)\exp[-(\eta_\nu - \bar\eta_\nu)]\frac{\bar r\,\mathrm{d}\bar r}{(\bar r^2 - r_o^2)^{\frac{1}{2}}} \end{aligned}\right\} \tag{4}$$

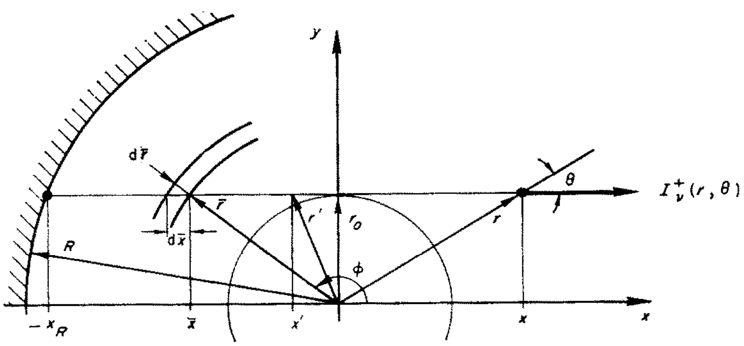

FIG. 1. Definition of $I_\nu^+(r,\theta)$ in the spherically symmetric atmosphere.

It also follows that $I_\nu^-(r, \theta)$ takes the form

$$I_\nu^-(r, \theta) = I_\nu^-(R, \theta) \exp[-(\eta_{\nu R} - \eta_\nu)] + \int_r^R \varkappa_\nu(\bar{r}) B_\nu(\bar{r}) \exp[-(\bar{\eta}_\nu - \eta_\nu)] \frac{\bar{r}\, d\bar{r}}{(\bar{r}^2 - r_o^2)^{\frac{1}{2}}} \quad (5)$$

where

$$\eta_\nu(r, r_o) = \int_{r_o}^{r} \frac{r' \varkappa_\nu(r')\, dr'}{(r'^2 - r_o^2)^{\frac{1}{2}}};$$

$$\eta_{\nu R} \equiv \bar{\eta}_\nu(R;\, r_o), \quad \bar{\eta}_\nu \equiv \eta_\nu(\bar{r};\, r_o)$$

and

$$r_o = r \sin \theta$$

These equations represent general expressions for the intensity in a spherically symmetric atmosphere, which will be used in the present study.

RATE OF ABSORPTION AND EMISSION IN THE ATMOSPHERE

Consider next the rate of absorption at the point r due to the surrounding atmosphere. The rate of absorption A_ν per unit volume at a point is given by

$$A_\nu = \varkappa_\nu \int_0^{4\pi} I_\nu\, d\Omega \quad (6)$$

where the integration extends over all of the elements of solid angle $d\Omega$ about the point in question. Accordingly, a spherical polar coordinate system is centered on the point r, and the element of solid angle can be written as $d\Omega = 2\pi \sin\theta\, d\theta$. It follows that A_ν can be expressed as

$$A_\nu = 2\pi\varkappa_\nu \int_0^{\pi/2} [I_\nu^+(r, \theta) + I_\nu^-(r, \theta)] \sin\theta\, d\theta \quad (7)$$

The rate of emission E_ν per unit volume at a point for an atmosphere in local thermodynamic equilibrium is expressed by

$$E_\nu = 4\pi\varkappa_\nu B_\nu(T) \quad (8)$$

The equation governing the radiative transport is then

$$Q = \int_0^\infty (A_\nu - E_\nu)\, d\nu = 0 \quad (9)$$

where Q is the net rate of heat input to the gas per unit volume as the result of radiation. Conservation of energy requires this quantity to be zero in the assumed absence of convection and molecular transport phenomena.

Finally, we observe that, if the boundary of the atmosphere is a black wall, the intensities $I_\nu^+(R, \theta)$ and $I_\nu^-(R, \theta)$ are given by

$$I_\nu^+(R, \theta) = I_\nu^-(R, \theta) = B_\nu(T_O)$$

where T_O is the uniform temperature of the black sphere surface with radius R.

EQUATION FOR TEMPERATURE DISTRIBUTION BETWEEN TWO CONCENTRIC SPHERES

Consider two black concentric spheres with radii R_1 and R_2, having constant and uniform but unequal temperatures T_1 and T_2, respectively. We wish to determine the temperature distribution in the gas between the two spheres as a function of radius vector r. To simplify the problem, it will be assumed at the outset that the absorption coefficient of the gas is a constant independent of the frequency and thermal state. Actually, we could conceivably take $\varkappa_\nu = ar^n$, where a and n are constants, and try different values of n. However, $n = 0$ appears to give the simplest formulation of the problem, which will suffice for the present purpose.

By applying the previous formalism and the assumption that the absorption coefficient is a constant $\varkappa$, we find, with the additional aid of Stefan's law, that the transfer equation of the problem can be written in the form of a Fredholm integral equation of the second kind for the unknown function T^4:

$$4\xi T^4 = T_1^4[\phi_1(\xi_1, \xi) + \phi_3(\xi_1, \xi)] + T_2^4[\phi_2(\xi_2, \xi) - \phi_3(\xi_2, \xi)] + 2\int_{\xi_1}^{\xi_2} T^4 \bar{\xi}\{E_1(|\xi - \bar{\xi}|) - E_1[\sqrt{(\xi^2 - \xi_1^2)} + \sqrt{(\bar{\xi}^2 - \xi_1^2)}]\}\, d\bar{\xi} \quad (10)$$

where $\xi = \varkappa r$, $\xi_1 = \varkappa R_1$ and $\xi_2 = \varkappa R_2$. Furthermore ϕ_1, ϕ_2 and ϕ_3 are the functions

$$\phi_1(\bar{\xi}, \xi) = (\xi + \bar{\xi})\, E_2(\xi - \bar{\xi}) - \exp[-(\xi - \bar{\xi})]$$

$$\phi_2(\bar{\xi}, \xi) = (\xi + \bar{\xi})\, E_2(\bar{\xi} - \xi) + \exp[-(\bar{\xi} - \xi)]$$

$$\phi_3(\bar{\xi}, \xi) = [\sqrt{(\bar{\xi}^2 - \xi_1^2)} - \sqrt{(\xi^2 - \xi_1^2)}]\, E_2[\sqrt{(\xi^2 - \xi_1^2)} + \sqrt{(\bar{\xi}^2 - \xi_1^2)}] + \exp\{-[\sqrt{(\bar{\xi}^2 - \xi_1^2)} + \sqrt{(\xi^2 - \xi_1^2)}]\}$$

and $E_1(t)$ and $E_2(t)$ are special cases of the integro-exponential function $E_n(t)$ of general order n (for the definition, see reference [1] or [2]). The term on the left-hand side of equation (10) represents the total rate of emission of the element of gas at the point ξ, the first two terms on the right-hand side designate surface contributions, and the integral term represents the contribution from the gas to the total rate of absorption of the element of gas at the point ξ. Because of the nonlinear form of the arguments in this integral equation, there is little hope of an exact analytical solution. The kernel function possesses a logarithmic singularity at the point $\bar{\xi} = \xi$; therefore, the equation as it stands is unsuited for a numerical study. The singular behavior of the kernel can be changed, however, by performing an integration by parts under the integral sign. In so doing, it is to be observed that the gas temperature immediately adjacent to the two spherical surfaces is not in general equal to the surface temperatures themselves in the assumed absence of molecular transport phenomena, i.e. $T_{\xi=\xi_1} \neq T_1$, $T_{\xi=\xi_2} \neq T_2$. The integration by parts reduces then the integral equation of the second kind to one of first kind for the function $\mathrm{d}T^4/\mathrm{d}\xi$, i.e.

$$0 = (T_1^4 - T_{\xi=\xi_1}^4)\,[\phi_1(\xi_1, \xi) + \phi_3(\xi_1, \xi)]$$
$$+ (T_2^4 - T_{\xi=\xi_2}^4)\,[\phi_2(\xi_2, \xi) - \phi_3(\xi_2, \xi)]$$
$$- \int_{\xi_1}^{\xi_2} (\mathrm{d}T^4/\mathrm{d}\bar{\xi})\, K(\bar{\xi};\ \xi, \xi_1)\, \mathrm{d}\bar{\xi} \qquad (11)$$

where

$$K(\bar{\xi}; \xi, \xi_1) = \operatorname{sgn}(\xi - \bar{\xi})(\xi + \bar{\xi})\, E_2(|\xi - \bar{\xi}|)$$
$$- \exp[-(|\xi - \bar{\xi}|)] + [\sqrt{(\bar{\xi}^2 - \xi_1^2)}$$
$$- \sqrt{(\xi^2 - \xi_1^2)}]\, E_2[\sqrt{(\bar{\xi}^2 - \xi_1^2)}$$
$$+ \sqrt{(\xi^2 - \xi_1^2)}] + \exp\{-[\sqrt{(\bar{\xi}^2 - \xi_1^2)}$$
$$+ \sqrt{(\xi^2 - \xi_1^2)}]\} \qquad (12)$$

The kernel of equation (11) has a finite jump at the point $\bar{\xi} = \xi$. This equation is now suited for numerical study; results of such a study are described subsequently.

It is clearly seen that equation (11) is satisfied identically in the case when the temperature is everywhere the same. Also, we note that equation (10) agrees with a similar equation derived in reference [8] by another method if $T_1 = T_2$ in equation (10) and if the heat generation term included in reference [8] is added to equation (10). Inspection of equation (10) or (11) shows furthermore, that if we are interested in the temperature T itself as a function of ξ, three parameters govern the solution, namely ξ_1, ξ_2 and T_1/T_2. However, the temperature ratio of the surfaces does not enter as a parameter, if one chooses to consider the function

$$(T^4 - T_1^4)/(T_1^4 - T_1^4).$$

In terms of this function equation (10) can be written as

$$\frac{T^4 - T_1^4}{T_2^4 - T_1^4} = \frac{1}{4\xi}\{\phi_2(\xi_2, \xi) - \phi_3(\xi_2, \xi)\}$$
$$+ \frac{1}{2\xi}\int_{\xi_1}^{\xi_2} \frac{T^4 - T_1^4}{T_2^4 - T_1^4}\,\bar{\xi}\{E_1(|\xi - \bar{\xi}|)$$
$$- E_1[\sqrt{(\xi^2 - \xi_1^2)} + \sqrt{(\bar{\xi}^2 - \xi_1^2)}]\}\,\mathrm{d}\bar{\xi} \qquad (13)$$

Equation (13) shows explicitly that only two parameters, ξ_1 and ξ_2, influence the solution for the function $(T^4 - T_1^4)/(T_2^4 - T_1^4)$. In this paper we choose to discuss the temperature distribution itself rather than the function

$$(T^4 - T_1^4)/(T_2^4 - T_1^4).$$

RATE OF HEAT TRANSFER

Along with the temperature distribution in the gas it is of considerable interest to determine the net heat-transfer rate q_ν per unit frequency through an element of area with given orientation within the gas. If we choose the normal of the surface element in question to have the (outward) direction of radius vector in the atmosphere, q_ν can be written as

$$q_\nu = 2\pi \int_0^{\pi/2} I_\nu^+ \cos\theta \sin\theta\, \mathrm{d}\theta$$
$$- 2\pi \int_0^{\pi/2} I_\nu^- \cos\theta \sin\theta\, \mathrm{d}\theta \qquad (14)$$

In this formula the first term represents the heat flow going in the positive outward direction of radius vector, the second term the heat flow going in the opposite direction. By applying equation (14) to the present problem, one obtains the following formula for q (the integral of q_ν over frequency):

$$
\left.\begin{aligned}
q(\xi) = {} & \tfrac{1}{2}\,\sigma T_1^4 \xi^{-2}\,\{(\xi+\xi_1)^2 E_3(\xi-\xi_1) \\
& - (\xi^2-\xi_1^2)E_3\,\sqrt{(\xi^2-\xi_1^2)} \\
& + [1+\sqrt{(\xi^2-\xi_1^2)}]\exp[-\sqrt{(\xi^2-\xi_1^2)}] \\
& - (1+\xi-\xi_1)\exp[-(\xi-\xi_1)]\} \\
& + \tfrac{1}{2}\,\sigma T_2^4\xi^{-2}\,\big[[\sqrt{(\xi_2^2-\xi_1^2)}-\sqrt{(\xi^2-\xi_1^2)}]^2 \\
& E_3[\sqrt{(\xi_2^2-\xi_1^2)}+\sqrt{(\xi^2-\xi_1^2)}] \\
& - (\xi_2+\xi)^2 E_3(\xi_2-\xi) - [1+\sqrt{(\xi_2^2-\xi_1^2)} \\
& + \sqrt{(\xi^2-\xi_1^2)}]\exp\{-[\sqrt{(\xi_2^2-\xi_1^2)} \\
& + \sqrt{(\xi^2-\xi_1^2)}]\} + (1+\xi_2-\xi) \\
& \exp[-(\xi_2-\xi)]\big] \\
& + \frac{\sigma}{\xi^2}\int_{\xi_1}^{\xi_2} \bar{\xi}T^4 L(\bar{\xi};\xi,\xi_1)\,\mathrm{d}\bar{\xi}
\end{aligned}\right\} \quad (15)
$$

where $L(\bar{\xi};\xi,\xi_1)$ is given by

$$
\left.\begin{aligned}
L(\bar{\xi};\xi,\xi_1) = {} & \operatorname{sgn}(\xi-\bar{\xi})(\xi+\bar{\xi}) \\
& E_2(|\xi-\bar{\xi}|) + \exp[-(|\xi-\bar{\xi}|)] \\
& + [\sqrt{(\bar{\xi}^2-\xi_1^2)} - \sqrt{(\xi^2-\xi_1^2)}] \\
& E_2[\sqrt{(\bar{\xi}^2-\xi_1^2)} + \sqrt{(\xi^2-\xi_1^2)}] \\
& - \exp[-(\sqrt{(\bar{\xi}^2-\xi_1^2)} + \sqrt{(\xi^2-\xi_1^2)})]
\end{aligned}\right\} \quad (16)
$$

This result is a consequence of the previous formulas, because the heat flux q is related to the heat input function Q defined by equation (9) as follows

$$
-\frac{\varkappa}{\xi^2}\frac{\partial}{\partial\xi}(q\xi^2) = Q = A_{\text{tot}} - E_{\text{tot}} \qquad (17)
$$

Since equation (15) involves an integral over the temperature distribution, the heat flux q is given at any point ξ once the temperature as a function of ξ is known. To calculate q it is therefore necessary to solve equation (10) or its equivalent [equation (11)] first. However, equation (10) is obtained by setting Q equal to zero. Hence, it follows from equation (17) that

$$
\xi^2 q(\xi) = \text{const.} \qquad (18)
$$

The simplest form of the constant in equation (18) is obtained by letting $\xi = \xi_1$ in equation (15) and by rewriting the integral in terms of the function $(T^4 - T_1^4)/(T_2^4 - T_1^4)$. The resulting formula can be reduced to the following form

$$
\left.\begin{aligned}
\frac{\xi_1^2 q(\xi_1)}{\sigma(T_2^4 - T_1^4)} = {} & -\tfrac{1}{2}\{3\sqrt{(\xi_2^2-\xi_1^2)} \\
& E_4\sqrt{(\xi_2^2-\xi_1^2)} + (\xi_2+\xi_1)^2 E_3(\xi_2-\xi_1) \\
& + \exp[-\sqrt{(\xi_2^2-\xi_1^2)}] - (1+\xi_2-\xi_1) \\
& \exp[-(\xi_2-\xi_1)]\} \\
& - 2\int_{\xi_1}^{\xi_2} \frac{T^4 - T_1^4}{T_2^4 - T_1^4}[\xi_1 E_2(\xi-\xi_1) \\
& + E_3\sqrt{(\xi^2-\xi_1^2)} - E_3(\xi-\xi_1)]\xi\,\mathrm{d}\xi
\end{aligned}\right\} \quad (19)
$$

Since the function $(T^4 - T_1^4)/(T_2^4 - T_1^4)$ contains only ξ_1 and ξ_2 as parameters it is seen that the heat-transfer rate constant written in this form depends only on ξ_1 and ξ_2.

LIMITING SOLUTIONS

Exact solutions of the equation determining the temperature distribution between the two spheres can only be obtained by numerical methods. However, two *exact* limiting solutions corresponding to optically thin ($\varkappa \to 0$) and optically thick ($\varkappa \to \infty$) conditions can be obtained in closed form.

(a) *Optically thin gas*

The temperature distribution between the two spheres is obtained directly from equation (10) by performing the limiting process $\varkappa \to 0$. This leads to the result

$$
\frac{T}{T_2} = \left[\left(\frac{T_1}{T_2}\right)^4 \frac{\xi - \sqrt{(\xi^2-\xi_1^2)}}{2\xi} + \frac{\xi + \sqrt{(\xi^2-\xi_1^2)}}{2\xi}\right]^{\frac{1}{4}} \qquad (20)
$$

This solution is tantamount to a vanishing gas absorption. Note that the formula predicts an infinite temperature gradient at $\xi = \xi_1$ when $T_1 \neq T_2$, but a zero gradient when $T_1 = T_2$.

Also by taking the limit of equation (19) as $\varkappa \to 0$, one finds that the heat-transfer rate constant is

$$
\xi_1^2\, q(\xi_1)/\sigma(T_1^4 - T_2^4) = \xi_1^2 \qquad (21)
$$

(b) *Optically thick gas*

For the case of an optically thick gas, the rate of heat transfer q is given by the formula [10]

$$q(\xi) = -\tfrac{4}{3}\frac{d}{d\xi}(\sigma T^4) \qquad (22)$$

From equation (18) it follows then that

$$\frac{d}{d\xi}(\sigma T^4) = \frac{A}{\xi^2}$$

where A is an integration constant. In the limit as $\varkappa \to \infty$ it can be shown readily from equation (10) that the boundary conditions are

$$\varkappa \to \infty: \quad T = T_1 \quad \text{at} \quad \xi = \xi_1,$$

and

$$T = T_2 \quad \text{at} \quad \xi = \xi_2.$$

By applying these boundary conditions we obtain the following exact closed form solution for the temperature distribution:

$$\frac{T}{T_2} = \left[\left(\frac{T_1}{T_2}\right)^4 \frac{\xi_1(\xi_2 - \xi)}{\xi(\xi_2 - \xi_1)} + \frac{\xi_2(\xi - \xi_1)}{\xi(\xi_2 - \xi_1)}\right]^{\frac{1}{4}} \qquad (23)$$

Accordingly, the heat-transfer rate constant is

$$\xi_1^2\, q(\xi_1)/\sigma(T_1^4 - T_2^4) = 4\xi_1\xi_2/3(\xi_2 - \xi_1) \qquad (24)$$

(c) *Reduction to the flat plate case*

The theory developed here contains the plane-parallel configuration as a special case. To obtain the formulas pertaining to this case, we substitute $\xi_2 = \xi_1 + L$ and

$$\xi = \xi_1 + \chi\,(0 \leqslant \chi \leqslant L)$$

into the equations and let $\xi_1 \to \infty$; L is the optical distance between the plates and χ is the variable optical distance. For example, by making the indicated substitution into equations (20) and (23) and taking the limit as $\xi_1 \to \infty$, we obtain, after some rearrangement, the well-known formulas for this configuration

$$\text{Thin Gas:} \quad \frac{T^4 - T_1^4}{T_2^4 - T_1^4} = \tfrac{1}{2};$$

$$\text{Thick Gas:} \quad \frac{T^4 - T_1^4}{T_2^4 - T_1^4} = \frac{\chi}{L}$$

PRESENTATION OF RESULTS

In the numerical part of the study, the method of undetermined coefficients was selected for solving equation (11), and the equation was accordingly programmed and a number of different cases computed on an IBM 7094. In all cases presented here, the given interval $\xi_2 - \xi_1$ was divided into a maximum of 109 equal subintervals. To test convergence of the solution as affected by the number of subintervals, the problems were also solved using 82 and 55 subintervals. From these additional calculations it was concluded that the accuracy in all the presented temperature distributions is better than $\Delta T/T_2 = 0{\cdot}002$.

Three parameters govern the exact solution for the temperature distribution itself, namely, the temperature ratio T_1/T_2 and the optical radii ξ_1 and ξ_2. In order to exhibit some of the pertinent parametric effects, sets of temperature profiles with different wall temperature ratios are presented: $T_1/T_2 = 2$ (Fig. 2), $T_1/T_2 = 5$ (Fig. 3), and $T_1/T_2 = 25$ (Fig. 4). In each figure exact (numerical) solutions are displayed as unbroken lines, and the thin and thick gas limiting solutions according to equations (20) and (23) are shown as broken lines. The different solutions shown for each combination of ξ_1 and ξ_2 are to be interpreted as follows. The temperature curves marked ξ correspond directly to the ξ-values shown on the ξ-scale. Thus, for these solutions ξ_2 is held fixed at $\xi_2 = 10$, while ξ_1 varies and takes on the values $\xi_1 = 5$, 1, 0·1. This curve sequence can be explained as a result of varying R_1 but keeping R_2 and the absorption coefficient $\varkappa$ fixed. Then, to see what happens if $\varkappa$ is reduced (or increased) by a factor of ten in the cases just described, the curves marked $0{\cdot}1\xi$ (10ξ) are shown. In reducing (increasing) $\varkappa$ by a factor of ten, the indicated values on the ξ-scale are reduced (increased) by a factor of ten, such that ξ_2 for this sequence of exact curves is held fixed at the value $\xi_2 = 1$ (100), whereas ξ_1 takes on the values $\xi_1 = 0{\cdot}5$, 0·1, 0·01 (50, 10, 1). The limiting solutions are, by definition, independent of $\varkappa$ and remain unchanged when the ξ-scale is multiplied by factors of ten. For each fixed geometrical arrangement the different curves provide a direct indication of how the solution of equation (11) is affected by changes in the absorption coefficient. Figures 2–4 thus describe the effects of the temperature ratio, the geometry, and the absorption coefficient on the solution.

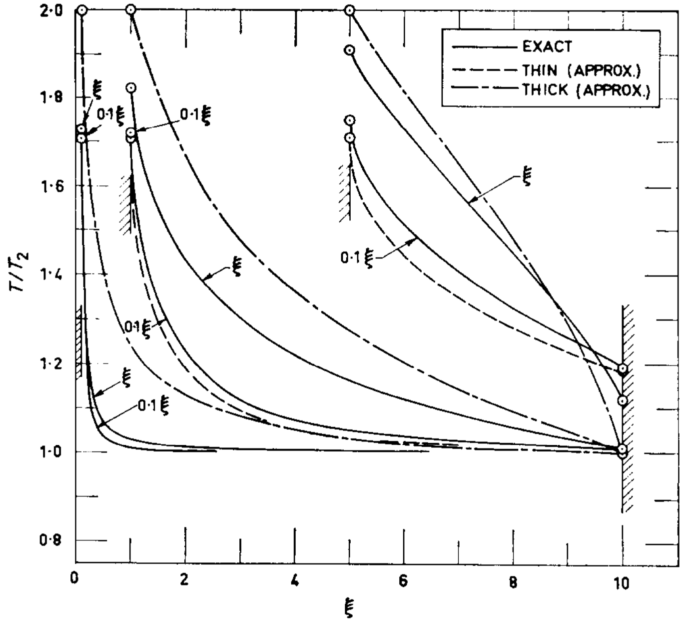

FIG. 2. Temperature curves for $T_1/T_2 = 2$.

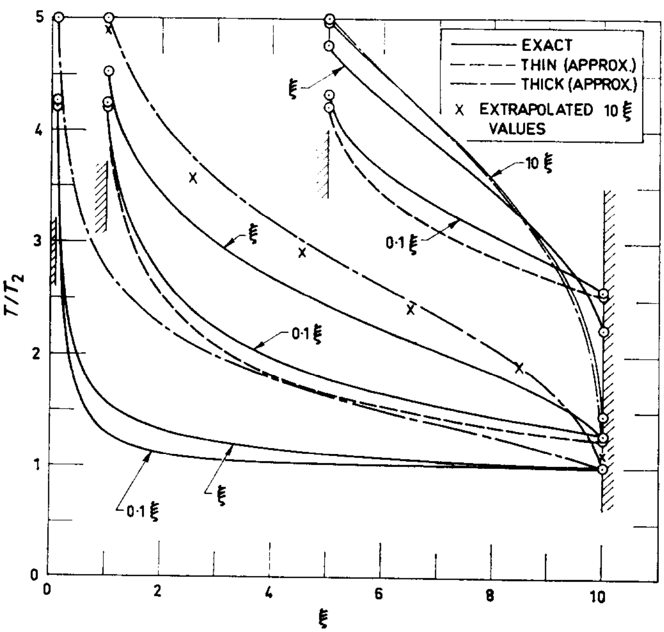

FIG. 3. Temperature curves for $T_1/T_2 = 5$.

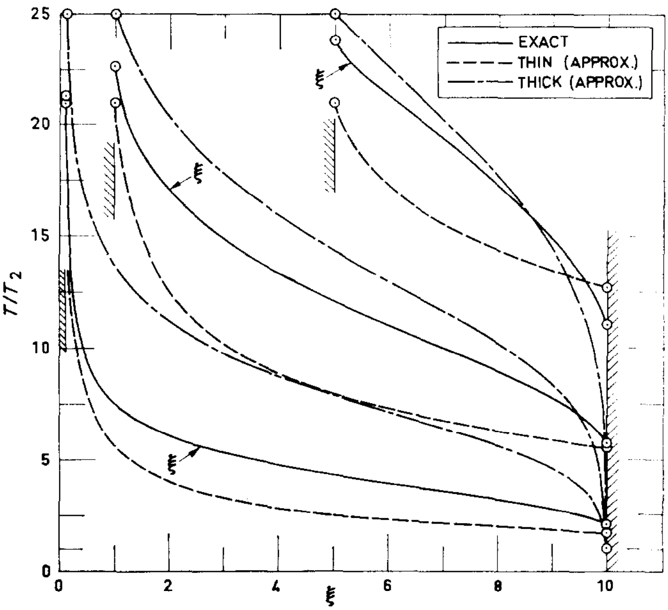

FIG. 4. Temperature curves for $T_1/T_2 = 25$.

The attempts to calculate the 10ξ curves were not always successful because, when the interval $\xi_2 - \xi_1$ is large, the maximum number of subintervals possible in the program is in general not sufficiently large to ensure accurate data. In Fig. 3 one accurate 10ξ curve is displayed; extrapolated 10ξ values for the intermediate case ($\xi_2 = 100$, $\xi_1 = 10$) are shown. The extrapolated values were obtained from three different calculations of the same profile using a different number of subintervals (109, 82 and 55) and the assumption, in analysing the data, that the convergence to the correct temperature at a fixed ξ is exponential with respect to the number of subintervals.

In Fig. 4 only one set of numerical solutions are shown. It is felt that in this case the assumption of a constant absorption coefficient is strongly limited, and the curves are only shown as an indication of trends.

DISCUSSION AND CONCLUSIONS

In studying the results presented in the previous section, we notice first the apparent effect that the optical size of the inner and hotter sphere has on the solution. When ξ_1 is small compared to ξ_2, the gas temperature decreases rapidly within a narrow region close to the inner sphere, resembling a boundary-layer effect. Concurrent with the rapid variation in the temperature close to the inner sphere is a large temperature slip at the inner surface, whereas a small slip is found at the outer surface. As the optical size of the inner sphere becomes larger, the gas temperature varies much more gradually between the two spheres, and the temperature slip at the outer sphere becomes appreciable. These general trends in the temperature curves can be explained as an effect of the spherical symmetry. On the one hand, when ξ_1 is small compared to ξ_2, we have a case that can be represented by a radiative heat source of small but finite dimensions inside a large spherical container. The temperature varies rapidly only near the source, and the temperature slip is consequently substantial at the inner surface. On the other hand, when ξ_1 and ξ_2 are nearly equal, a nearly plane parallel case is obtained. The trends in the temperature curves for these cases can be understood by studying previous investigations dealing with the flat plate configuration [3–6].

Next, we shall study the effect the absorption coefficient has on any particular solution with fixed values of R_1 and R_2. In the least conspicuous case, when ξ_2 and ξ_1 are relatively close together ($\xi_2/\xi_1 = 2$), the 10ξ curve is rather close to the limiting thick gas curve, whereas the $0{\cdot}1\xi$ curve does not come as close to the limiting thin gas curve (see Figs. 2 and 3). The exact numerical solutions demonstrate in these cases that the limiting thick and the thin gas solutions differ roughly by three orders of magnitude in the value of the absorption coefficient. Similar results are obtained in the flat plate case.

As we move to higher ratios of ξ_2/ξ_1, the calculated exact solutions tend to agree very well with the thin gas approximation, especially when the temperature ratio is low, see Fig. 2. Inspection of the thin gas approximation formula, equation (2), reveals, that the temperature distribution is independent of ξ_2. Therefore, as soon as $\xi \gg \xi_1$, the temperature is close to the asymptotic value $T/T_2 = 1$. Thus the location of the outer sphere surface plays no role for the temperature profile in these cases, in agreement with the concept of a radiative heat source inside a large spherical container. For a small but finite value of $\varkappa$, the absorption by the large body of gas which separates the small hotter sphere from the large cooler sphere, does not seem to have any appreciable effect on the exact curves in these cases.

The calculated temperature distributions can be used to calculate the heat-transfer rate constant, and Table 1 shows the value of $\xi_1^2 q(\xi_1)/\sigma(T_1^4 - T_2^4)$ for the various cases. The values of the heat-transfer rate constants in the limiting cases are not given, since very little meaning can be attached to these values.

Table 1. The heat-transfer rate constant $\xi_1{}^2 q(\xi_1)/\sigma(T_1{}^4 - T_2{}^4)$ for various cases

ξ_2	ξ_1	$\xi_1{}^2 q(\xi_1)/\sigma(T_1{}^4 - T_2{}^4)$ Exact numerical
10	0·1	$9{\cdot}28 \times 10^{-3}$
	1	$6{\cdot}65 \times 10^{-1}$
	5	9·46
1	0·01	$9{\cdot}92 \times 10^{-5}$
	0·1	$9{\cdot}65 \times 10^{-3}$
	0·5	$2{\cdot}24 \times 10^{-1}$

It has not been intended here to develop an approximate theory for the correlation of the data presented. The object has been rather to present exact solutions of the transfer equation in spherical symmetry, which has been obtained by numerical and limiting processes. It is, however, possible to develop an approximate theory on the basis of the diffusion approximation and include temperature jumps at the boundaries. The general background for such a theory is contained in a paper by Deissler [11]. Another way of building an approximate theory would be to use the theory of matched asymptotic expansions. The latter way is probably the most appropriate since the approximation can then be carried to any desired degree of accuracy, at least in principle.

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Résumé—Le transport d'énergie sous forme de rayonnement dans une atmosphère à symétrie sphérique est discuté. Comme application, on a étudié le transport entre deux sphères concentriques (possédant des surfaces noires) maintenues, à des températures uniformes et différentes et séparées par un gaz absorbant et émetteur avec un coefficient d'absorption constant. La distribution radiale de la température du gaz satisfait à une équation intégrale de Fredholm de seconde espèce. Des solutions exactes sont obtenues par une méthode numérique, et les solutions limites correspndant aux conditions d'une atmosphère optiquement épaisse et optiquement fine sont données sous forme analytique. La distribution radiale de la température dépend fortement du rapport des températures des deux surfaces, du coefficient d'absorption et des rayons de sphères.

Zusammenfassung—Der Austausch von Stahlungsenergie in einer Atmosphäre mit Kugelsymmetrie wird diskutiert. Als Anwendungsbeispiel wird der Austausch zwischen zwei konzentrischen Kugeln (mit schwarzen Oberflächen) studiert, die auf verschiedenen, gleichmässigen Temperaturen gehalten sind und deren Zwischenraum von einem absorbierenden und emittierenden Gas mit konstantem Absorptionskoeffizienten erfüllt ist. Die Gastemperatur entlang eines Halbmessers genügt einer Fredholm Integralgleichung zweiter Art. Exakte Lösungen wurden mit einer numerischen Methode erhalten und Grenzlösungen für optische dicke und dünne Bedingungen sind in geschlossener Form angegeben. Die Temperaturverteilung entland eines Halbmessers hängt stark vom Temperaturverhältnis der beiden Oberflächen, dem Absorptionskoeffizienten und den Halbmessern der Kugeln ab.

Аннотация—Рассматривается перенос лучистой энергии в атмосфере, имеющей сферическую симметрию. В качестве примера исследуется обмен между двумя концентрическими сферами с черными поверхностями, имеющими различную равномерную температуру, между которыми находится поглощающий и излучающий газ с постоянным коэффициентом поглощения. Распределение температуры газа вдоль радиуса удовлетворительно описывается интегральным уравнением Фредгольма второго рода. Численными методами получены точные решения. В замкнутой форме найдены решения для предельных случаев оптически толстой и тонкой среды. Температурное распределение вдоль радиуса сильно зависит от отношения температур на поверхности сфер, коэффициента поглощения и радиусов сфер.